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TORAL ACTIONS ON 4-MANIFOLDS AND THEIR CLASSIFICATIONS

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ABSTRACT. The existence of a cross-section is proved for some nonorientable 4-manifolds with a T^2-action. Two 4-manifolds with a T^2-action, which have the same previously known invariants, are constructed. By using a new homotopy invariant, they are proved to be homotopy inequivalent. Finally a stable diffeomorphism theorem is proved.

1. INTRODUCTION

P. Orlik and F. Raymond showed, in [OR, I], the following:

Suppose that M *is a 4-dimensional closed simply-connected manifold with an effective* T^2*-action. Then* M *is an equivariant connected sum of* $\mathbb{C}\mathrm{P}^2$, $\overline{\mathbb{C}\mathrm{P}^2}$, $S^2 \times S^2$ *and* S^4.

In [OR, II], they studied some non-simply-connected manifolds with an effective T^2-action and proved that, *if the manifolds have neither fixed points nor circle subgroups as stabilizers, then, in "almost all" cases, two manifolds are diffeomorphic if and only if they are equivariantly diffeomorphic up to an automorphism of* T^2.

With the presence of fixed points or circle orbits, the techniques of a topological classification are quite different. Orlik and Raymond obtained an equivariant classification when M has a fixed point, but there were three families of T^2-manifolds, called basic blocks. To obtain a topological classification of closed orientable T^2-manifolds with a fixed point, it was necessary to study these families of T^2-manifolds, which are described in terms of orbit spaces. For example, each of the manifolds of one family has the orbit space pictured in Figure 1 (see §2 for a description of this space and [OR, II]). They showed that

$$\begin{aligned} M\#k(S^2 \times S^2) &= (S^1 \times S^3)\#(S^2 \times S^2)\#k(S^2 \times S^2), && \text{if } mn \text{ is even}, \\ &= (S^1 \times S^3)\#\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}\#k(S^2 \times S^2), && \text{if } mn \text{ is odd}, \end{aligned}$$

where k is an integer. Whether or not the addition of the k copies of $S^2 \times S^2$ could be dropped was left unsettled.

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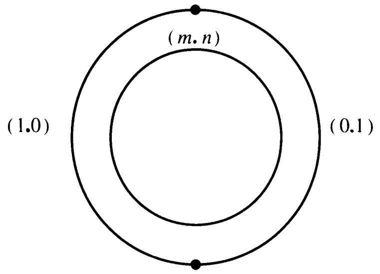


FIGURE 1. 2-dimensional annulus

P. Pao attacked these unsettled problems with elementary, but clever techniques and eliminated the necessity of stabilization of the above as well as the other two families in [P]. Furthermore, he proved the following theorem:

Suppose M is a 4-dimensional orientable closed manifold with an effective T^2-action. If M^* has a fixed point, then M can be decomposed into a connected sum of copies of S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$, $S^1 \times S^3$, L_n and L'_n ($n \geq 2$, integer) (for L_n and L'_n, see §4).

Unfortunately, this connected sum decomposition is not unique. For example, $\mathbb{C}P^2\#(S^2 \times S^2) = \mathbb{C}P^2\#\overline{\mathbb{C}P}^2\#\mathbb{C}P^2$. So Pao defined a normal form called a *normal decomposition*, of this connected sum decomposition. He proved the following topological classification theorem: *Every 4-dimensional orientable closed manifold with a T^2-action has a unique normal decomposition if its orbit space has a fixed point.*

Because the classification of closed orientable 4-manifolds which admit an effective T^2-action is almost complete, it is natural to ask whether there exist similar results for nonorientable 4-manifolds. Furthermore, nonorientable 4-manifolds have not been studied as extensively as the orientable ones, and there are not many concrete examples of them in the literature. By studying nonorientable 4-manifolds with a T^2-action we are able to give concrete realizations by means of orbit spaces and orbit structures. The orbit space is a 2-manifold with boundary to which are attached certain "weights" which enable us to reconstruct the 4-manifold. While this is not a cell decomposition, it behaves like one in that the manifold is divided into nice pieces which we can topologically identify. We then compute the corresponding attaching maps and attempt to topologically identify and classify the resulting 4-manifolds.

First, we study the problem of classifying nonorientable 4-dimensional manifolds up to equivariant diffeomorphism in §2. The key to this is the existence of a cross-section (2.6). This enables us to reconstruct the 4-manifold in certain basic cases from the weighted orbit space. This is a 2-manifold with boundary weighted by the orbit invariants of the action.

In §3, we investigate simple T^2-manifolds whose orbit spaces are of the following type. (See Figure 2.) They admit cross-sections to the orbit mapping. The topological identification of these T^2-manifolds proves to be very difficult. We use well-known invariants to distinguish all but a pair of them. The final equivariantly inequivalent pair cannot be distinguished by any of the previously

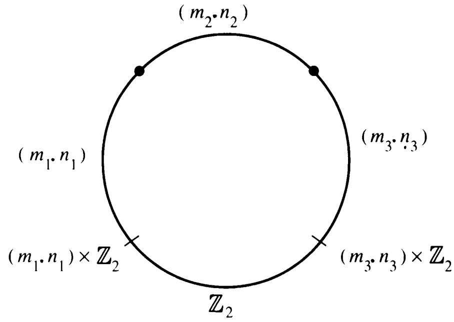


FIGURE 2

known homotopy type invariants. However, by using a new invariant, due to S. Kojima (see [KKR]), constructed for the purpose of distinguishing these manifolds, we are able to show they are not homotopy equivalent. In general, this invariant is hard to compute, but, because we have such a nice description of our manifolds, we are able to give explicit geometric computations.

In §4, we turn to the problem of topologically identifying the nonorientable closed T^2-manifolds which admit fixed points, but no $\mathbb{Z}_2\times\mathbb{Z}_2$ stabilizers on the boundary of their orbit spaces. We show that, by adding connected sums of $\mathbb{C}P^2$'s, the resulting manifold can be decomposed into a connected sum of eight basic manifolds and two families of T^2-manifolds. We also show that this decomposition is far from unique.

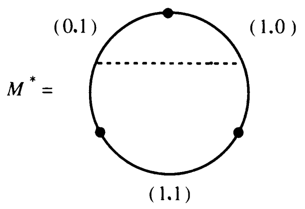
2. THE EXISTENCE OF A CROSS-SECTION AND THE EQUIVARIANT CLASSIFICATION THEOREM

Notation and definition. All manifolds are nonorientable 4-dimensional smooth manifolds, unless specified otherwise. $I=[0\ 1]$. $S^1=$ the set of all complex numbers whose absolute value is 1. $R=$ all real numbers. $\mathbb{C}=$ all complex number $\mathbb{C}^2=$ the 2-dimensional complex plane. $R^2=$ 2-dimensional Euclidean space. $T^2=S^1\times S^1$ (2-dimensional torus).

(2.1) **Definition.** Let f be a function from R^2 to $\mathbb{C}^2$ defined by $f(x.y)=(\exp(2\pi ix), \exp(2\pi iy))$, where $\exp(2\pi it)=\cos 2\pi t+\imath\sin 2\pi t$, t a real number. Given relatively prime integers m, n, we define the image of the straight line $mx+ny=0$ in R^2 under f to be $(m.n)$.

(2.2) **Definition.** A T^2-action on M is effective if $gx=x$, for all x in M, implies $g=e$ (i.e. identity) in T^2. In this case, we call M a T^2-manifold. We denote the quotient space (or orbit space) by M^* (i.e. M/T^2), and the natural projection from M to M^* is π.

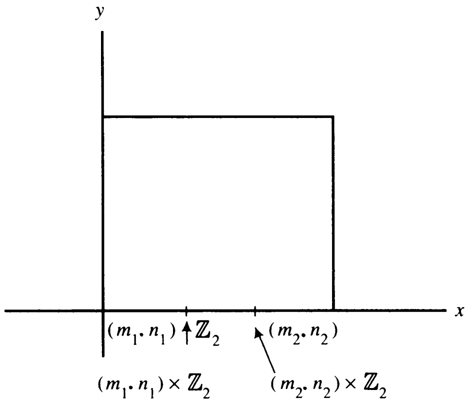
(2.3) **Definition.** Weighted orbit space. Let M be a T^2-manifold. Using the slice theorem, we find that M^* is a 2-dimensional manifold. By assigning the stabilizer subgroups of T^2 to orbits in M^* as "weights", we may speak of M^* as the weighted orbit space (briefly, orbit space). For example,



is a weighted disk. The weight at each interior point is the identity, while we have divided up the boundary into three arcs. The three end points of the arcs correspond to fixed points and the interior of the arcs correspond to orbits whose stabilizers are (0.1), (1.1), (1.0) (see [OR, I] for more details). According to [OR, I], the preimage of the dotted line is an invariant S^3, the portion of the disk above the line is an invariant 4-dimensional disk and the portion of the disk below the line is an invariant disk fiber bundle over S^2 with a structure group S^1 in M. Therefore, we can see that M is $\mathbb{C}\mathrm{P}^2$.

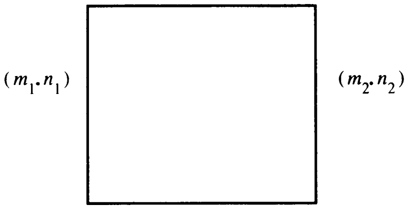
We begin with some preliminary steps for the proof of a cross-section theorem, which will be crucial in the Equivariant Classification Theorem (2.7).

(2.4) **Lemma.** *Let* $\pi: M \to M^*$ *be as above. Suppose that* $M^* (= I \times I)$ *is given as below*:



More precisely, on the set $\{(x, 0): 0 \leq x < 1/3\}$, *the stabilizer is* $(m_1.n_1)$. *On the set* $\{(x, 0): 1/3 < x < 2/3\}$, *the stabilizer is a* $\mathbb{Z}_2$ *subgroup of* T^2. *On the set* $\{(x, 0): 2/3 < x < 1\}$, *the stabilizer is* $(m_2.n_2)$, *at* $(1/3, 0)$, *the stabilizer is* $(m_1.n_1) \times \mathbb{Z}_2$, *at* $(2/3, 0)$, *the stabilizer is* $(m_2.n_2) \times \mathbb{Z}_2$. *Otherwise the stabilizer is trivial. Then* π *has a cross-section* (*i.e. there exists a map* χ *such that* $\pi \circ \chi = \mathrm{Id}$). *Moreover, any cross-section* χ *on* $I \times \{1\} \cup \{0\} \times I \cup \{1\} \times I = A$ *may be extended to a cross-section over* M^*. *Note that* $\mathbb{Z}_2$ *is not contained in the circle subgroups.*

Proof. Since M^* has a $\mathbb{Z}_2$ stabilizer on the boundary, by considering the slice representation, M is nonorientable. Let $\widetilde{M}$ be the orientable double covering of M. Then, we have an induced T^2-action on M which commutes with the regular covering transformations (cf. [B, p. 66]). We have the induced $\mathbb{Z}_2$-action on $\widetilde{M}^*$ and the natural projection from $\widetilde{M}^*$ to M^*. In particular, $\widetilde{M}^* = I \times I$ is given as below:



where $A_1 = I \times \{1\}$. The induced $\mathbb{Z}_2$-action on $\widetilde{M}^*$ acts a rotation by 180 degrees or a reflection. Since M is nonorientable, it must act as a reflection. Therefore, there exists a cross-section χ_1 from M^* to $\widetilde{M}^*$. By [OR, I], there exists a cross-section χ_2 from $\widetilde{M}^*$ to $\widetilde{M}$ and $\widetilde{M} = L \times I$ where L is a lens space, whose exact form depends on $(m_1.n_1)$ and $(m_2.n_2)$.

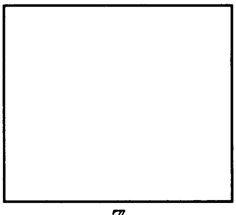
Consider the map $\rho \circ \chi_2 \circ \chi_1$ from M^* to M, where ρ is the natural projection from $\widetilde{M}$ to M. Note that $\rho^* \circ \tilde{\pi} = \pi \circ \rho$ where ρ^* is the natural projection from $\widetilde{M}^*$ to M^*, and $\tilde{\pi}$ is the natural projection from $\widetilde{M}$ to $\widetilde{M}^*$. Then $\pi \circ \rho \circ \chi_2 \circ \chi_1 = \rho^* \circ \tilde{\pi} \circ \chi_2 \circ \chi_1 = \rho^* \circ \mathrm{Id}_{\widetilde{M}^*} \circ \chi_1 = \rho \circ \chi_1 = \mathrm{Id}_{M^*}$.

So $\rho \circ \chi_2 \circ \chi_1$ is a cross-section to the orbit map π. Moreover, given a cross-section χ on A, by [OR, I], $\chi(A)$ is a lens space L in M.

Since L is orientable and $\rho\colon \widetilde{M} \to M$ is a double covering, $\rho^{-1}(\chi(A))$ is the disjoint union of two lens spaces, i.e. $\rho^{-1}(\chi(A)) = L \times \{0\} \cup L \times \{1\}$. We may assume that $\chi_2(A_1) = L \times \{0\}$. Then $\chi_1 \circ \chi(A) = L \times \{0\}$. Since $\rho^* \circ (\tilde{\pi} \circ \chi_1 \circ \chi) = \pi \circ \rho \circ \chi_1 \circ \chi = \pi \circ \mathrm{Id}_M \circ \chi = \mathrm{Id}_{M^*}$ $\tilde{\pi} \circ \chi_1 \circ \chi$ is a cross-section from A to $\widetilde{M}^*$ and $\tilde{\pi} \circ \chi_1 \circ \chi(A) = A_1$.

Let $\tilde{\chi} = \tilde{\pi} \circ \chi_1 \circ \chi$. We can extend $\tilde{\chi}$ to M^* in an obvious way. So $\rho \circ \chi_2 \circ \tilde{\chi}$ is an extension of χ. □

(2.5) **Lemma.** *Let $\pi\colon M \to M^*$ be as above. Suppose $M^* = I \times I$ is shown as below*:



More precisely, the stabilizer is $\mathbb{Z}_2$ on the $I \times \{0\}$, and elsewhere it is the identity. Then there exists a cross-section. Moreover, a given cross-section χ on $I \times \{1\} \cup \{0\} \times I \cup \{1\} \times I$ may be extended to M^*.

Proof. This is verified by a slight modification of Lemma (2.4). □

Now we are ready to state and prove the existence of a cross-section in more general cases.

(2.6) **Proposition.** *Let M be closed. Suppose M^* has a fixed point, has no $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizer on the boundary and no finite stabilizer in the interior. Then there exists a cross-section to the orbit map.*

Proof. As in Case 3 of Theorem 1.10 of [OR, I], consider a closed annular neighborhood of the boundary components. Denote their union by U^*. Note that, on $Y^* = M^* - U^*$, $\pi^{-1}(Y^*) \to Y^*$ is a T^2-principal bundle. By the classification of 2-dimensional manifolds with boundaries, Y^* is homotopy equivalent to $\bigvee S^1$ (a wedge product of S^1's).

Since $H^2(\bigvee S^1, \mathbb{Z} \times \mathbb{Z}) = H^2(Y^*, \mathbb{Z} \times \mathbb{Z}) = 0$, $\pi^{-1}(Y^*) = Y^* \times T^2$. This we may construct a cross-section over Y^* and, by applying Lemmas 1.6, 1.7, 1.8 of [OR, I] and previous lemmas, (2.4, 2.5), we get a global cross-section on M^*. □

(2.7) **Theorem** (Equivariant Classification Theorem). *Let M_1, M_2 be closed manifolds. Suppose that both M_1^* and M_2^* have a fixed point but have no $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizer on the boundaries. Then there exists a weight preserving diffeomorphism between M_1^* and M_2^*, if and only if there exists an equivariant diffeomorphism between M_1 and M_2.*

Remark. By the slice theorem, if there is an orbit in the interior of an orbit space whose stabilizer is nontrivial (i.e. other than identity), then the stabilizer is finite.

We can choose a closed disk in the interior of the orbit space. The preimage of the closed disk and the action is topologically equivalent to $(T^2, T^2 \times D^2/\mathbb{Z}_\alpha)$. Where $\mathbb{Z}_\alpha$ is the finite cyclic stabilizer subgroup of T^2, and the action of $\mathbb{Z}_\alpha$ on T^2 and D^2 is given as follows

$$\lambda \times z \to \lambda^\nu z, \qquad |z| \leq 1, \quad z \text{ in } D^2,$$

$$\lambda \times (z_1, z_2) \to (z_1 \lambda^{\gamma_1}, z_2 \lambda^{\gamma_2}), \qquad (z_1, z_2) \quad \text{in } T^2,$$

with $0 < \nu < \alpha$, α and ν are relatively prime, $\lambda = \exp(2\pi\imath/\alpha)$ and $0 \leq \gamma_1 < \alpha$, $0 \leq \gamma_2 < \alpha$. $T^2 \times D^2/\mathbb{Z}_\alpha$ means each point $((z_1, z_2), z)$ of $T^2 \times D^2$ is identified with $(\lambda(z_1, z_2), \lambda^{-1}z)$. The T^2-action on $T^2 \times D^2/\mathbb{Z}_\alpha$ is on the first coordinate by natural multiplication.

Proof. The argument for this theorem is really the same as Theorem 1.2 of [OR, I]. First, if M_1 and M_2 are equivariantly diffeomorphic, then this diffeomorphism induces a weight preserving map on the orbit spaces.

On the other hand, given such a diffeomorphism $h\colon M_1^* \to M_2^*$, we shall construct an equivariant diffeomorphism from M_1 to M_2. Choose closed 2-disks D about each x whose stabilizer is finite in the interior. They are mapped by h onto corresponding disks about $h(x_k^*)$. We can assume that x_k^* is in the interior of D_k^* which is in the interior of M_1^*, and that there is no other point in D_k^* whose stabilizer is nontrivial. If we let $M_{1.1}^* = M_1^* - \bigcup_{k=1}^n D_k^*$ and $M_{2.1}^* = M_2^* - \bigcup_{k=1}^n h(\overset{\circ}{D}{}_k^*)$, where $\overset{\circ}{D}{}_k^*$ is the interior of D_k^*, then $h_1\colon M_{1.1}^* \to M_{2.1}^*$, the restriction of h, is a weight preserving diffeomorphism.

Furthermore, as there are no finite stabilizers except the identity in the interior of $M_{1.1}$, any cross-section of the orbit map over $\bigcup_{k=1}^n \partial D_k^*$ can be extended to all of $M_{1.1}^*$ by Lemma (1.10) of [OR, I]. Thus it is easy to find an equivariant diffeomorphism $H\colon M_{1.1} \to M_{2.1}$.

To each $M_{\iota\cdot 1}$, $(\iota = 1, 2)$, we must attach the $(T^2, T^2 \times D_k/\mathbb{Z}_{\alpha,k})$ equivariantly, where $\mathbb{Z}_{\alpha,k}$ means $\mathbb{Z}_\alpha$ acting on D_k. We already have a cross section over each ∂D_k^*. We may attach $(T^2, T^2 \times D_k/\mathbb{Z}_{\alpha,k})$ up to equivariant diffeomorphism in only one way (up to equivariant isotopy) and thereby extend H to all of M_1. This completes the proof. □

3. SPECIAL CASES

Notation and definition. Let R^3 be 3-dimensional Euclidean space. Let ν denote a vector R^3. $S^2 = \{(x, y, z) \text{ in } R^3 \colon x^2 + y^2 + z^2 = 1\} = \{\nu \colon |\nu| = 1\}$. Since we may identify R^3 with $\mathbb{C} \times R$, S^2 may be expressed as follows: $S^2 = \{(\rho \exp(2\pi\iota\theta), z) \colon \rho \geq 0,\ \rho^2 + z^2 = 1\}$. RP^2 is real projective 2-space. We will use $[\nu]$ or $[(\rho\exp(2\pi\iota\theta), z)]$ respectively, to denote the point in RP^2 which is the image of ν or $(\rho\exp(2\pi\iota\theta), z)$ in S^2 under the natural projection from S^2 to RP^2. $\mathbb{C}\mathrm{P}^2$ is complex projective 2-space. $S^4 =$ the unit 4-sphere and $RP^4 =$ the real projective 4-space. D^2 is the unit disk in R^2, i.e. $\{(x \cdot y) \colon x^2 + y^2 \leq 1\} = \{r\exp(2\pi\iota\phi) \colon 0 \leq r \leq 1,\ \phi \in R\}$. $S^1 \times S^2/\simeq$ is the nontrivial S^2 bundle over the circle (i.e. the 3-dimensional nonorientable handle. Here $\simeq$ means every (α, ν) in $S^1 \times S^2$ is identified with $(-\alpha, -\nu)$, $\alpha \in \mathbb{C}$.

In this section, we shall first describe some special orbit spaces, then construct manifolds with T^2-actions, whose orbit spaces are the given ones. Then we will give the topological classification of these manifolds. (In fact, two distinct such manifolds are not even homotopy equivalent).

Now suppose that we have orbit spaces as shown in Figure 3 where $\mathbb{Z}_2$ is a subgroup of order 2 in T^2.

Recall from [OR, I] that, as the T^2-action is smooth and effective, determinants

$$\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} m_2 & n_2 \\ m_3 & n_3 \end{vmatrix}$$

must be 1 or -1.

By using an automorphism of T^2 (i.e. by reparametrizing T^2), we may assume that M^* is one of the following four types, where n, m, n', m' are

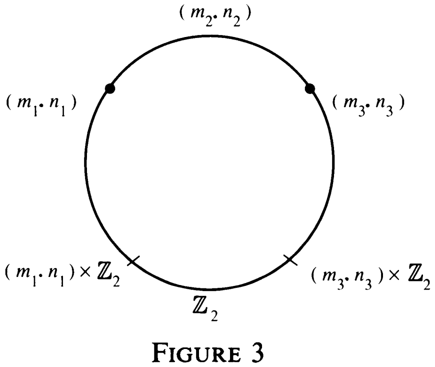
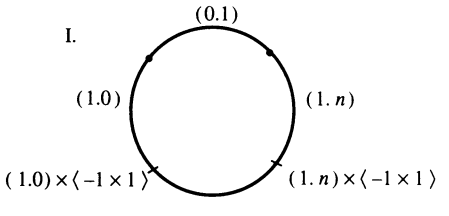


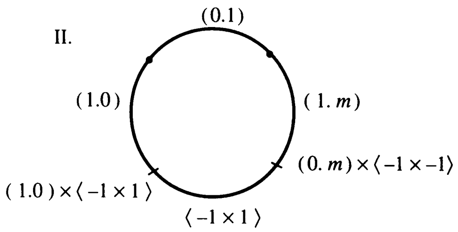
FIGURE 3

integers. In type II and type IV, note that m, m' must be even, because effectiveness implies $(1.m)$, $(-1.m')$ cannot contain the subgroup $\langle -1\times -1\rangle$ of order 2 generated by $-1\times -1\in S^1\times S^1$.

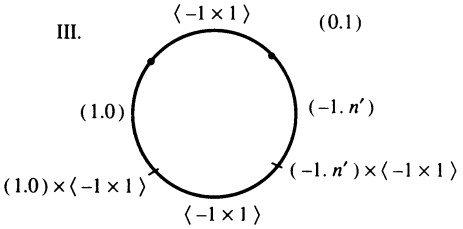
I.



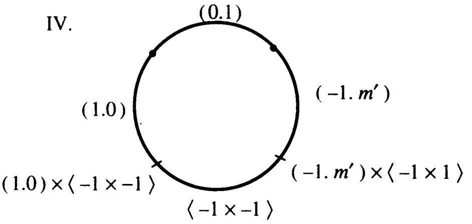
II.



III.



IV.



(3.2) **Theorem.** *Let M^* be of type* I *or type* III. *Then M is diffeomorphic to $RP^2\times S^2$ or $\mathbb{C}P^2\# RP^4$ according as n is even or odd.*

Remark. If M is a nonorientable manifold and X is oriented, then $M\# X$ is diffeomorphic to $M\#\overline{X}$, where $\overline{X}$ denotes X with the reverse orientation. So $\mathbb{C}P^2\# RP^4$ is diffeomorphic to $\overline{\mathbb{C}P^2}\# RP^4$ (see [H]).

To prove (3.2), we need the following lemma.

(3.3) **Lemma.** *Let f_n be a diffeomorphism from $S^1\times RP^2$ to $S^1\times RP^2$ defined as follows:*

$$(\exp(2\pi\imath\phi),\ [(r\exp(2\pi\imath\theta),\ z)])$$
$$\to (\exp(2\pi\imath\phi),\ [(r\exp 2\pi\imath(\theta+n\phi),\ z)]).$$

Then, if n is even, f_n can be extended to $D^2\times RP^2$ as a self-diffeomorphism. □

Proof. f_n may be written in the following way:

$$(\exp(2\pi\imath\phi),\ [\nu])\to\left(\exp(2\pi\imath\phi),\ \begin{pmatrix}\cos 2\pi n\phi & \sin 2\pi n\phi & 0\\ -\sin 2\pi n\phi & \cos 2\pi n\phi & 0\\ 0 & 0 & 1\end{pmatrix}(\nu)\right).$$

Then the map $\tilde{f}_n$ from S^1 to $SO(3)$ defined by

$$\exp(2\pi\imath\phi)\to\begin{pmatrix}\cos 2\pi n\phi & \sin 2\pi n\phi & 0\\ -\sin 2\pi n\phi & \cos 2\pi n\phi & 0\\ 0 & 0 & 1\end{pmatrix}$$

is a loop in $SO(3)$, so $\tilde{f}_n$ represents an element in $\Pi_1(SO(3))$. Since n is even, $\tilde{f}_n$ is homotopic to the trivial map i.e. there exists a homotopy H_n from

$S^1 \times I$ to $SO(3)$ such that

$$H_n(\exp(2\pi\imath\phi), 1) = \tilde{f}_n(\exp(2\pi\imath\phi)),$$

$$H_n(\exp(2\pi\imath\phi), 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

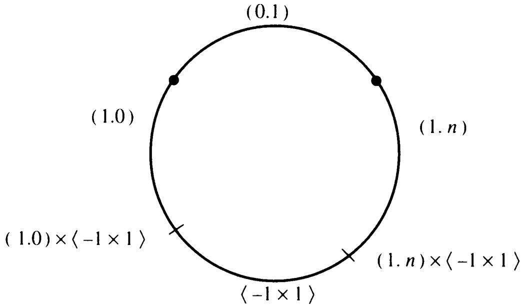
Define F_n from $D^2 \times RP^2$ to $D^2 \times RP^2$ by

$$(r\exp(2\pi\imath\phi), [\nu]) \to (r\exp(2\pi\imath\phi), \ [H_n(\exp(2\pi\imath\phi), r)[\nu])$$

Then since $F_n(r\exp(2\pi\imath\phi), [-\nu]) = (r\exp(2\pi\imath\phi), [H_n(\exp(2\pi\imath\phi), r)(-\nu)]) = F_n(r\exp(2\pi\imath\phi), [\nu])$, F_n is a well-defined diffeomorphism. This completes the proof. □

Let us return to (3.2).

Proof of Theorem (3.2). First, we will construct a manifold with a T^2-action whose orbit space is

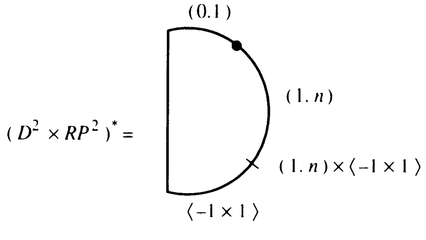


Consider a T^2-action on $D^2 \times RP^2$ as follows:

$$T^2 \times D^2 \times RP^2 \to D^2 \times RP^2,$$

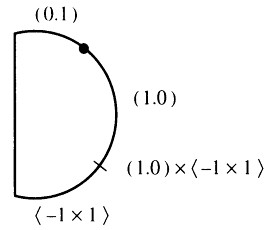
$$(\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta))(r\exp(2\pi\imath\phi), [\rho\exp(2\pi\imath\theta), z]) \\ \to (r\exp 2\pi\imath(\phi+\beta), [\rho\exp 2\pi\imath(\alpha + n\beta + \theta), z]).$$

Then,



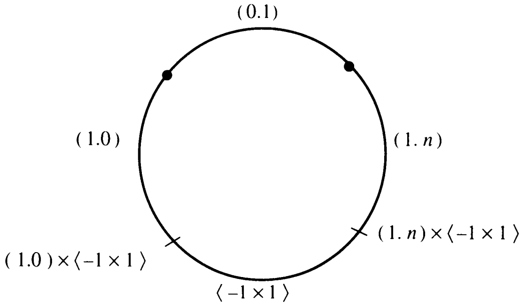
In particular, when $n = 0$,

$(D^2 \times RP^2)^* =$



If we glue two copies of $D^2 \times RP^2$ along the boundary by using the map f_n in (3.3), we get a 4-manifold $D^2 \times RP^2 \cup_{f_n} D^2 \times RP^2$. On the first copy, we give the action above when $n = 0$, and on the second copy, we give the action for general n. Then we can see that f_n is an T^2-equivariant diffeomorphism from $\partial D^2 \times S^2$ to $\partial D^2 \times S^2$.

Thus we have an 4-manifold $D^2 \times RP^2 \cup_{f_n} D^2 \times RP^2$ with a T^2-action whose orbit space is



Now we can complete the proof of the theorem as follows.

(i) *n is even.* Define a map $\mathrm{Id} * F_n$ from $D^2 \times RP^2 \cup_{\mathrm{id}} D^2 \times RP^2$ to $D^2 \times RP^2 \cup_{f_n} D^2 \times RP^2$ as follows:

On the first copy of $D^2 \times RP^2$,

$$(r\exp(2\pi\imath\phi), [\nu]) \to (r\exp(2\pi\imath\phi), [\nu])$$

on the second copy of $D^2 \times RP^2$,

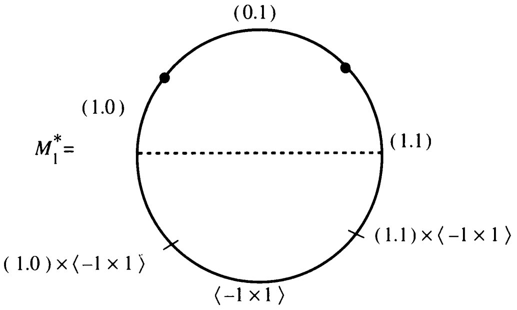
$$(r\exp(2\pi\imath\phi), [\nu]) \to (r\exp(2\pi\imath\phi), [H_n(\exp(2\pi\imath\phi), r)(\nu)]).$$

Then $\mathrm{Id} * F_n$ is a well-defined diffeomorphism.

(ii) *n is odd.* $n - 1$ is even, so we can apply (3.3). Define $\mathrm{Id} * F_{n-1}$, as in the case (i), from $D^2 \times RP^2 \cup_{f_1} D^2 \times RP^2$ to $D^2 \times RP^2 \cup_{f_n} D^2 \times RP^2$. Then $\mathrm{Id} * F_{n-1}$ is a well-defined diffeomorphism.

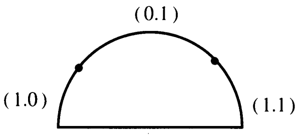
If we put $M_0 = D^2 \times RP^2 \cup_{\mathrm{id}} D^2 \times RP^2$, and $M_1 = D^2 \times RP^2 \cup_{f_1} D^2 \times RP^2$, $M_0 = S^2 \times RP^2$, and we see that M_1 is the nontrivial RP^2-fiber bundle over S^2 with the structure group S^1. Furthermore, consider the dotted line in M_1^*,

$M_1^* =$

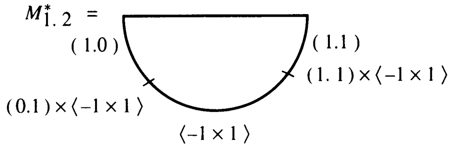


The preimage in M_1 of the line is a T^2-invariant S^3. If we cut M_1 along that S^3, we have $M_{1.1}$ and $M_{1.2}$ whose orbit spaces are

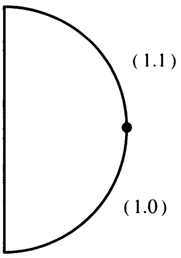
$M_{1.1}^* =$



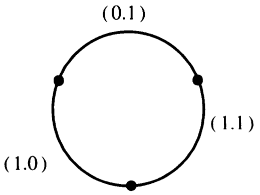
$M_{1.2}^* =$

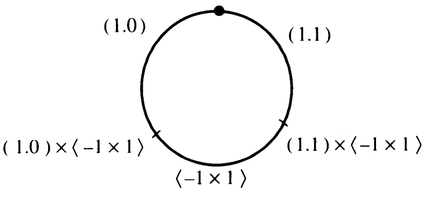


By adding two copies of



whose total space is a 4-dimensional disk, to $M_{1.1}^*$ and $M_{1.2}^*$, we have





In other words, to total spaces $M_{1.1}$ and $M_{1.2}$, we have to attach 4-dimensional disks along the boundaries equivariantly.

This procedure implies that M_1 is a connected sum of two manifolds. By [OR, I], a manifold with the first orbit space is diffeomorphic to $\mathbb{C}P^2$. For the second orbit space, we give a T_2-action on RP^4 as follows:

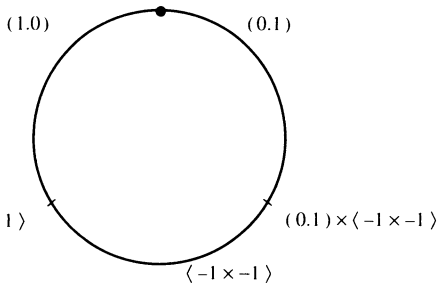
Since

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

commutes with

$$\begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta & 0 & 0 & 0 \\ -\sin 2\pi\theta & \cos 2\pi\theta & 0 & 0 & 0 \\ 0 & 0 & \cos 2\pi\phi & \sin 2\pi\phi & 0 \\ 0 & 0 & -\sin 2\pi\phi & \cos 2\pi\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

we can give a $\mathbb{Z}_2 \times T^2$-action on S^4 naturally and also give a T^2-action on RP^4, there is only one orbit space up to automorphism of T^2. Thus the manifold whose orbit space is

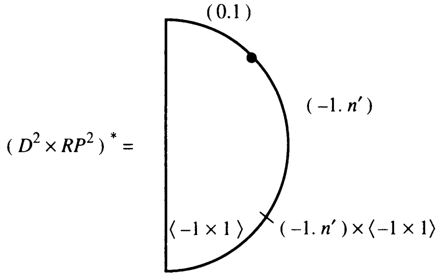


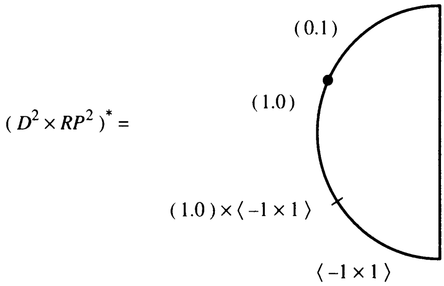
is RP^4. We conclude that any manifold M of type I is diffeomorphic to $RP^4 \# \mathbb{C}P^2$.

In the same way, by giving appropriate T^2-actions on two copies of $D^2 \times RP^2$ and gluing the copies along the boundaries, we can construct a manifold whose orbit space is of type III as follows:

$$\begin{gathered} T^2 \times D^2 \times RP^2 \to D^2 \times RP^2, \\ (\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta)) \times (r\exp(2\pi\imath\phi), [\rho\exp(2\pi\imath\theta), z]) \\ \to (r\exp 2\pi\imath(\phi+\beta), [\rho\exp 2\pi\imath(-\alpha+n'\beta+\theta), z]), \\ T^2 \times D^2 \times RP^2 \to D^2 \times RP^2, \\ (\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta)) \times (r\exp(2\pi\imath\phi), [\rho\exp(2\pi\imath\theta), z]) \\ \to (r\exp 2\pi\imath(\phi+\beta), [\rho\exp 2\pi\imath(\alpha+\theta), z]). \end{gathered}$$

The orbit spaces are as follows:



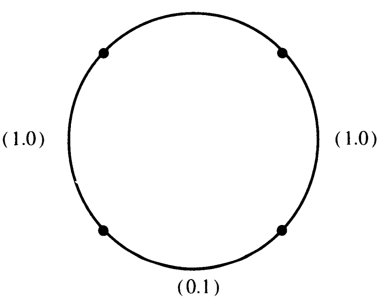


Let f_n' be a self-diffeomorphism of $\partial D^2 \times RP^2$ defined by

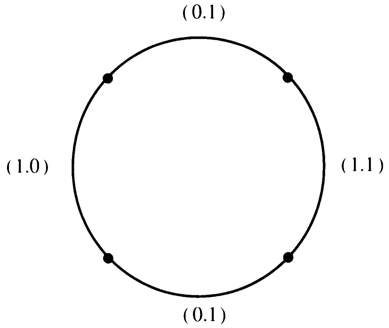
$$(\exp(2\pi\imath\phi), [\rho \exp(2\pi\imath\theta), z]) \to (\exp(2\pi\imath\phi), [\rho \exp 2\pi\imath(-\theta + n'\phi), z]).$$

We can see that the map is T^2-equivariant with respect to the above two actions. Since the set of self-diffeomorphisms of RP^2 is path-connected, the map, when $n' = 0$, is isotopic to identity. So we can apply the procedure in type I. This completes the proof. □

Remark. For the above two spaces, by considering double coverings with their induced T^2-actions, we can see that their orbit spaces are



and



respectively.

So $RP^4\#\mathbb{C}P^2$ has $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ as its orientable double covering and $S^2 \times RP^2$ has $S^2 \times S^2$ as its orientable double covering by [OR, I]. Therefore $RP^4\#\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \times RP^2$.

(3.4) **Theorem.** *Let* M^* *be of type* II *or type* IV. *Then* M *is diffeomorphic to* $S^2 \times S^2/\simeq$ *or* $D^2 \times S^2/\simeq \cup_{g_2} D^2 \times S^2/\simeq$.

In the first case, the equivalence relation $\simeq$ means $((\rho\exp(2\pi\imath\phi), z, \nu)$ is identified with $((-\rho\exp(2\pi i\phi), z), -\nu)$. In the second case, $\simeq$ means $(r\exp(2\pi\imath\phi), \nu)$ is identified with $(-r\exp(2\pi\imath\phi), -\nu)$. The gluing map g_2 of two $D^2 \times S^2/\simeq$ from $\partial D^2 \times S^2/\simeq$ to itself defined by

$$[(\exp(2\pi\imath\phi), \nu)] \to [(\exp(2\pi\imath\phi), \tilde{f}_2(\exp(2\pi\imath\phi))(\nu))],$$

$\tilde{f}_2$ is the map in (3.3), according as $m = 4j$ or $m = 4j + 2$, j integer. As in (3.2), we need the following lemma.

(3.5) **Lemma.** *Let* g_m *be a self-diffeomorphism from* $S^1 \times S^2/\simeq$ *to itself, defined by*

$$[(\exp(2\pi\imath\phi), \nu)] \to [(\exp(2\pi\imath\phi), \tilde{f}_m(\exp(2\pi\imath\phi))(\nu))]$$

Then, if $m = 4j$, g_m *can be extended to* $D^2 \times S^2/\simeq$ *where* $\tilde{f}_m$ *is the map in* (3.3).

Proof. Recall from (3.3) that the map $\tilde{f}_m \colon S^1 \to SO(3)$ is given by

$$\exp(2\pi\imath\phi) \to \begin{pmatrix} \cos 2\pi m\phi & \sin 2\pi m\phi & 0 \\ -\sin 2\pi m\phi & \cos 2\pi m\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$H_m \colon S^1 \times I \to SO(3),$$

$$H_m(\exp(2\pi\imath\phi), 1) = \tilde{f}_m(\exp(2\pi\imath\phi)),$$

$$H_m(\exp(2\pi\imath\phi), 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define a map $\widetilde{G}_{2j}$ from $D^2 \times S^2/\simeq$ to itself by

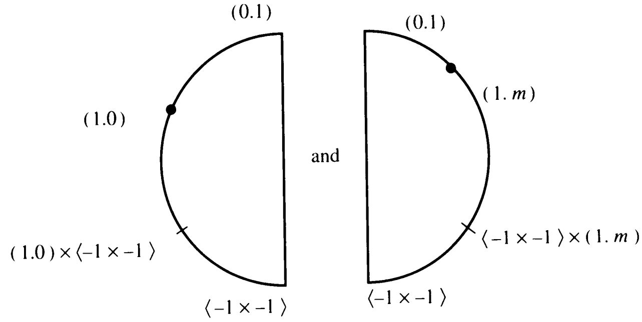
$$[(r\exp(2\pi\imath\phi), \nu)] \to [(r\exp(2\pi\imath\phi), H_{2j}(\exp(2\pi\imath\phi, r))(\nu))].$$

Then, since $\tilde{f}_{2j}(\exp(2\pi\imath\phi)) = \tilde{f}_{4j}(\exp(2\pi\imath\phi))$, $\widetilde{G}_{2j}$ is a well-defined diffeomorphism. This completes the proof. □

Proof of (3.4). As in the proof of (3.2), we shall construct a manifold whose orbit space is of type II, and get the same results on type IV. Consider T^2-actions on $D^2 \times S^2/\simeq$ as follows:

$$\begin{gathered}
T^2 \times D^2 \times S^2/\simeq \to D^2 \times S^2/\simeq, \\
(\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta)) \times [(r\exp(2\pi\imath\phi), \{\rho\exp(2\pi\imath\theta), z\})] \\
\to [(r\exp 2\pi\imath(\phi+\beta), \{\rho\exp 2\pi\imath(\theta+\alpha+m\beta), z\})], \\
T^2 \times D^2 \times S^2/\simeq \to D^2 \times S^2/\simeq, \\
(\exp(2\pi\imath\alpha), \exp(2\pi\imath\beta)) \times [(r\exp(2\pi\imath\phi), \{\rho\exp(2\pi\imath\theta), z\})] \\
\to [(r\exp 2\pi\imath(\phi+\beta), \{\rho\exp 2\pi\imath(\theta+\alpha+m\beta), z\})].
\end{gathered}$$

Then, its orbit spaces are respectively,



It is easy to check that g_m in (3.5) is a T^2-equivariant (with respect to the above two actions) self-diffeomorphism of $D^2 \times S^2/\simeq$. So by gluing two copies of $D^2 \times S^2/\simeq$ along the boundary through g_m, we have a T^2-manifold $D^2 \times S^2/\simeq \cup_{g_m} D^2 \times S^2/\simeq$

(i) $m = 4j$. Define an $\mathrm{Id} * \widetilde{G}_{2j}$ as follows:

$$D^2 \times S^2/\simeq \cup_{\mathrm{Id}} D^2 \times S^2/\simeq \to D^2 \times S^2/\simeq \cup_{g_{4j}} D^2 \times S^2/\simeq$$

on the first copy of $D^2 \times S^2/\simeq$

$$[(r\exp(2\pi\imath\phi), \nu)] \to [(r\exp(2\pi\imath\phi), \nu)]$$

on the second copy of $D^2 \times S^2/\simeq$

$$[(r\exp(2\pi\imath\phi), \nu] \to \widetilde{G}_{2j}([(r\exp(2\pi\imath\phi), \nu)]).$$

Then, $\mathrm{Id} * \widetilde{G}_{2j}$ is a well-defined diffeomorphism, since

$$\widetilde{G}_{2j}[(-r\exp(2\pi\imath\phi), -\nu)] = -\widetilde{G}_{2j}[(r\exp(2\pi\imath\phi), \nu)].$$

(ii) $m = 4j+2$. We have a well-defined diffeomorphism $\mathrm{Id} * \widetilde{G}_{2j}$, as in the case (i), from $D^2 \times S^2/\simeq \cup_{g_2} D^2 \times S^2/\simeq$ to $D^2 \times S^2/\simeq \cup_{g_{4j+2}} D^2 \times S^2/\simeq$. Note that $D^2 \times S^2/\simeq \cup_{\mathrm{id}} D^2 \times S^2/\simeq = S^2 \times S^2/\simeq$. Similarly, for type IV, we can make a construction analogous to type II. This completes the proof. □

Remark. $S^2 \times S^2/\simeq$ is an S^2-bundle over RP^2 with the structure group $\mathbb{Z}_2$. The self-diffeomorphism g_2 is not diffeotopic to the identity (see [KR]).

(3.6) **Theorem.** *$RP^2 \times S$ is not homotopy equivalent to $S^2 \times S^2/\simeq$.*

Proof. We want to show that all elements in $H^2(S^2 \times RP^2, \mathbb{Z}_2)$ have zero square, i.e. for every l in $H^2(S^2 \times RP^2, \mathbb{Z}_2)$, $l \cup l = l^2 = 0$ in $H^4(S^2 \times RP^2, \mathbb{Z}_2)$, but there is δ in $H^2(S^2 \times S^2/\simeq, \mathbb{Z}_2)$ such that δ^2 is not equal to 0. First, by the Künneth formula,

$$\begin{aligned} &H^2(RP^2 \times S^2, \mathbb{Z}_2) \\ &\qquad = H^2(RP^2, \mathbb{Z}_2) \otimes H^2(S^2, \mathbb{Z}) \oplus H^2(RP^2, \mathbb{Z}_2) \otimes H^2(S^2, \mathbb{Z}) \\ &\qquad = (\mathbb{Z}_2 \otimes \mathbb{Z}) \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \end{aligned}$$

If a is represented by $S^2 \times \{y_0\}$ where y_0 is a base point of RP^2, and b is represented by $\{x_0\} \times RP^2$, where x_0 is the north pole of S^2, then $a \cup a = a^2 = b \cup b = b^2 = 0$, since we can deform $\{x_0\} \times RP^2$, and $S^2 \times \{y_0\}$ slightly so that there is no self-intersection respectively. Since any element τ in $H^2(RP^2 \times S^2, \mathbb{Z}_2)$ can be expressed as $\tau = ma + nb$, where $0 \le m$, $n \le 1$, $\tau = (ma+nb)^2 = m^2a^2 + 2mnab + n^2b^2 = 2mnab = 0$, (the coefficient is $\mathbb{Z}_2$).

On the other hand,

$$\begin{gathered} RP^2 \overset{\mathrm{incl}}{\hookrightarrow} S^2 \times S^2/\simeq \xrightarrow{\mathrm{proj}} RP^2, \\ [\nu] \mapsto [(x, \nu)] \mapsto [\nu], \end{gathered}$$

where incl is the inclusion and proj is the natural projection. Then $\mathrm{proj} \circ \mathrm{incl} = \mathrm{Id}$,

$$H_2(RP^2, \mathbb{Z}_2) \xrightarrow{(\mathrm{incl})^*} H_2(S^2 \times S^2/\simeq, \mathbb{Z}_2) \xrightarrow{(\mathrm{proj})^*} H_2(RP^2, \mathbb{Z}_2)$$

and $(\mathrm{proj})_* \circ (\mathrm{incl})_* = \mathrm{Id}$. For $(\mathrm{incl})_*(\alpha)$ in $H_2(S^2 \times S^2/\simeq, \mathbb{Z}_2)$, by Poincaré duality, there is the corresponding class κ in $H^2(S^2 \times S^2/\simeq, \mathbb{Z}_2)$, where α in $H_2(RP^2, \mathbb{Z}_2)$ is the nontrivial element. We want to prove κ^2 is not zero. Since κ^2 is the geometric self-intersection number of $(\mathrm{incl})_*(\alpha)$ (mod 2), we have only to prove that the self-intersection number of the base space RP^2 of $S^2 \times S^2/\simeq$ is nonzero (mod 2). We want to deform the base space RP^2 to get the self-intersection number. So we have only to consider a tubular neighborhood of RP^2 which is diffeomorphic to $R^2 \times S^2/\simeq$, where $\simeq$ means $((x, y), \nu))$ is identified with $((-x, -y), -\nu))$. In other words, we have only to count the self-intersection numbers of RP^2 in $R^2 \times S^2/\simeq$. The self-intersection number can be computed as follows: Note that $R^2 \times S^2/\simeq = R \times S^2/\simeq \oplus R \times S^2/\simeq$ where $\oplus$ is the Whitney sum of the vector bundles and $\simeq$ in $R \times S^2$ means (x, ν) is identified with $(-x, -\nu)$. Since $R \times S^2/\simeq$ is the tautological bundle over RP^2 which is nontrivial (cf. [MS, p. 38]), if we let $R \times S^2/\simeq = \gamma_2^1$ the mod 2 self-intersection of the base space RP^2 is the second Stiefel-Whitney class of $R^2 \times S^2/\simeq$. The total class is

$$\begin{aligned} \omega(\gamma_2^1 \oplus \gamma_2^1) &= \omega(\gamma_2^1)\omega(\gamma_2^1) = (1 + \omega_1(\gamma_2^1))(1 + \omega_1(\gamma_2^1)) \\ &= 1 + 2\omega_1(\gamma_2^1) + \omega_1^2(\gamma_2^1). \end{aligned}$$

So

$$\omega_2(R^2 \times S^2/\simeq) = \omega_2(\gamma_2^1 \oplus \gamma_2^1) = \omega_1^2(\gamma_2^1).$$

Since $\omega_1^2(\gamma_2^1)$ is the nonzero element in $H^2(RP^2, \mathbb{Z}_2)$ (cf. [MS, p. 47]), $\omega_2(\gamma_2^1 \oplus \gamma_2^1) = 1$. This implies κ^2 is not zero (mod 2). □

(3.7) **Theorem.** *$S^2 \times S^2/\simeq$ is not homotopy equivalent to M_2 $(= D^2 \times S^2/\simeq \cup_{g_2} D^2 \times S^2/\simeq)$.*

To prove (3.7), we need a homotopy invariant which was discovered recently. In *Homotopy invariants of nonorientable* 4*-manifolds* (see [KKR]), the invariant was defined as follows: Choose an element x in $\Pi_2(M)$, and represent it by a transversally immersed 2-sphere S_x. Then fixing an orientation of $\widetilde{M}$ (the orientable double covering of M), define a function q with values in $\mathbb{Z}_4$ by

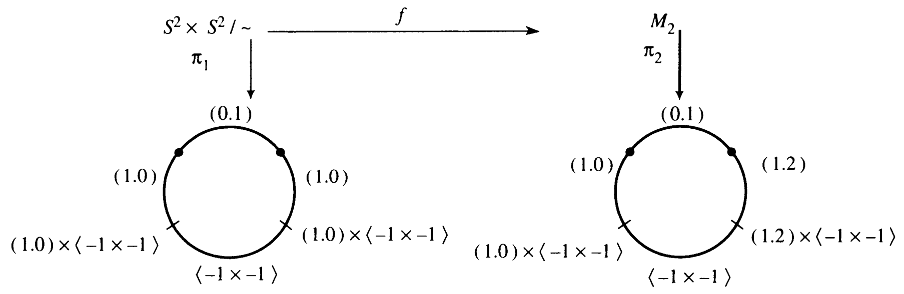
$$\mathscr{X}(\nu(S_x)) + 2\#\text{ self } S_x \pmod 4.$$

Here $\mathscr{X}(\nu(S_x))$ is Euler number of the pull-back bundle $f^*(\nu(S_x))$ with respect to the orientation induced from $\widetilde{M}$. Here f is a transversally immersed map from S^2 to M, $\nu(S_x)$ is a normal bundle over S_x in M and # self S_x means the self-intersection cardinal number S_x in M.

Now we are ready to state the main theorem in that paper without proof.

Theorem. *Suppose that f is a homotopy equivalence from M to N. Then $q_M(x) = q_N(f_*(x))$, for all x in $\Pi_2(M)$, with respect to relevant orientations of the orientable double covers M and N.*

Proof of (3.7). Suppose there exists a homotopy equivalence f from $S^2 \times S^2/\simeq$ to M_2. Then we have



where π_1, π_2 are the natural projections.

Consider the orientable double covering $S^2 \times S^2$ with the induced T^2-actions. Then we have orbit spaces as shown in Figure 4 corresponding to those shown in Figure 5 respectively. Choose an element x in $\Pi_2(S^2 \times S^2/\simeq)$ (the second homotopy group of $S^2 \times S^2/\simeq$) which corresponds to the preimage of the arc A (i.e. $\pi_1^{-1}(A)$) in $S^2 \times S^2/\simeq$. Then $q_M(x) = 0 + 0 = 0 \pmod 4$.

Next we want to show that $f_*(x)$ in $\Pi_2(M_2)$ is represented by $\pi_2^{-1}(B)$ of the arc B in Figure 6 let

$$U_1 = \{(x, y) \colon x^2 + y^2 \leq 1,\ x < 1/2\}$$

and

$$U_2 = \{(x, y) \colon x^2 + y^2 \leq 1,\ x > -1/2\}.$$

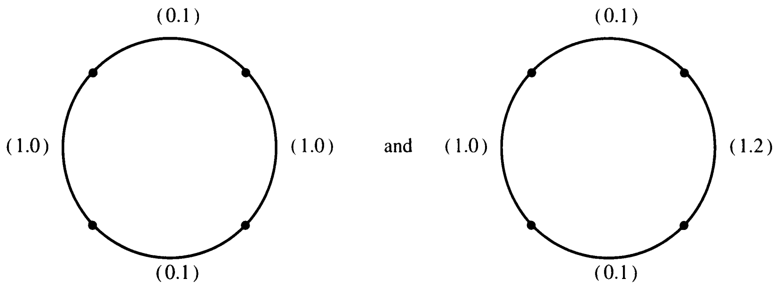


FIGURE 4

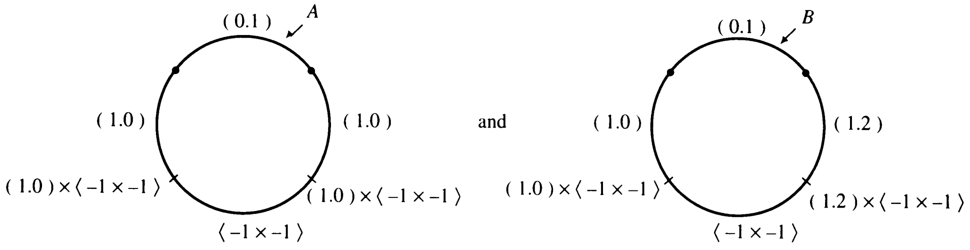


FIGURE 5

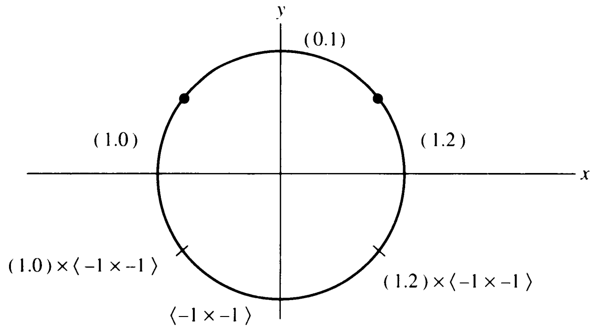


FIGURE 6

Note that $\pi_2^{-1}(U_1) = \pi_2^{-1}(U_2) = \mathring{D}^2 \times S^2/\simeq$ where $\mathring{D}^2 =$ the interior of D^2 and $\pi_2^{-1}(U_1 \cap U_2)$ is homotopy equivalent to $S^1 \times S^2/\simeq$ from (3.4). By the Mayer-Vietoris sequence, we have

$$
\begin{aligned}
&H_2(\pi_2^{-1}(U_1)) \oplus H_2(\pi_2^{-1}(U_2)) \to H_2(M_2) \xrightarrow{\delta_*} H_1(S^1 \times S^2/\simeq) \\
&\to H_1(\pi_2^{-1}(U_1)) \oplus H_1(\pi_2^{-1}(U_2)) \to H_1(M_2) \to 0
\end{aligned}
$$

where δ_* is the boundary homomorphism. Since $\pi_2^{-1}(U_1)$ and $\pi_2^{-1}(U_2)$ have the same homotopy type as RP^2 and the fundamental group of $S^1 \times S^2/\simeq$ is

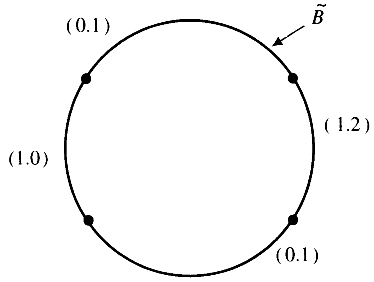
$\mathbb{Z}$, we obtain the exact sequence

$$0 \to H_2(M_2) \to \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H_1(M_2) \to 0.$$

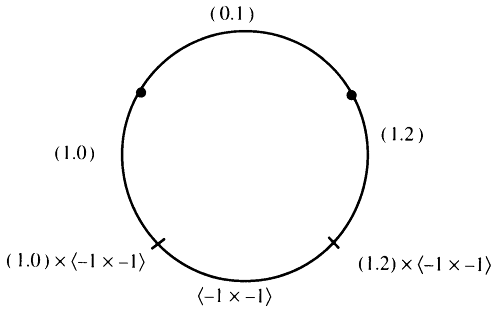
Since $H_1(M_2) = \mathbb{Z}$, $H_2(M_2)$ is a subgroup of $\mathbb{Z}$ and nonzero, so $H_2(M_2) = \mathbb{Z}$. The generator of $H_1(S^1 \times S^2/\simeq)$ is represented by $\pi_2^{-1}(P)$. By the definition of δ_*, $\pi_2^{-1}(B)$ represents the generator of $H_2(M_2)$. By the Hurewicz homomorphisms and their naturality, i.e.

$$\begin{tikzcd}
\Pi_2(S^2 \times S^2/\simeq) \arrow[r] \arrow[d, "f_*"', "\cong"] & H_2(S^2 \times S^2/\simeq) = \mathbb{Z} \arrow[d, "f_*"', "\cong"] \\
\Pi_2(M_2) \arrow[r] & H_2(M_2) = \mathbb{Z}
\end{tikzcd}$$

$f_*(x)$ in $\Pi_2(M_2)$ is represented by $\pi_2^{-1}(B)$, and $\pi_2^{-1}(B)$ corresponds to $\tilde{\pi}_2^{-1}(\widetilde{B})$ in $S^2 \times S^2$, where $\widetilde{B}$ is an arc as shown below:



According to [OR, I], the Euler number of the normal bundle of $\tilde{\pi}_2^{-1}(\widetilde{B})$ is the determinant $\left|\begin{smallmatrix}1&0\\1&2\end{smallmatrix}\right|$ with a sign. We have $\mathscr{X}(\nu(S_{f_*(x)})) = +2$ or -2. Since there is no self-intersection point in M as we can see

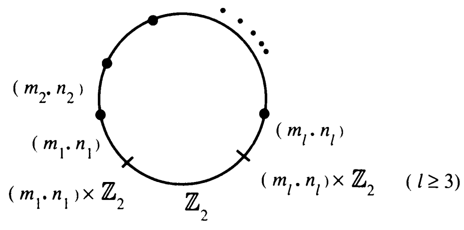


we have $q_{M_2}(f_*(x)) \equiv 2 \pmod 4$. We have a contradiction. We can conclude that those are not homotopy equivalent. □

4. STABLE DIFFEOMORPHISM THEOREM

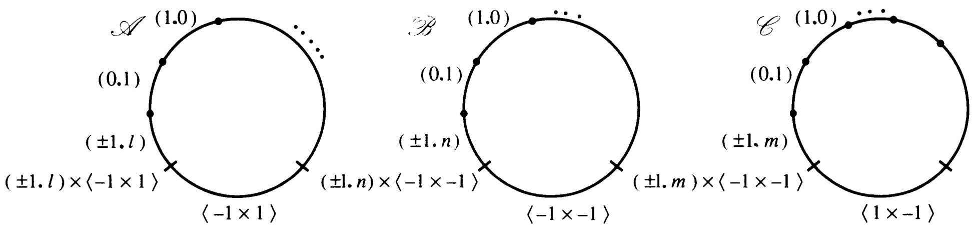
In this section, we are going to prove the stable diffeomorphism theorem (4.4). If M is closed, nonorientable and admits a toral action with fixed points and no $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizer, then M, after blowing up with $\mathbb{C}P^2$, s, is diffeomorphic to a connected sum of 8 basic T^2-manifolds and 2 families of T^2-manifolds. To prove the theorem, we need the following lemma:

(4.1) **Lemma.** *Suppose* M^* *is given as below* (*only one* $\mathbb{Z}_2$ *component*):



Then $M\#n\mathbb{C}P^2$ *can be expressed as a connected sum of* $S^2 \times S^2$, $\mathbb{C}P^2$, $RP^2 \times S^2$, *where* n *is some integer.*

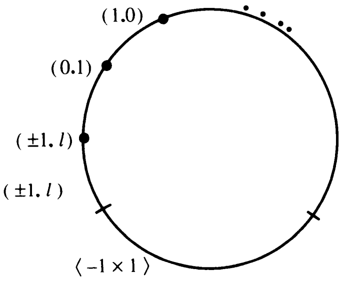
Proof. By reparametrizing T^2, we may assume that M^* is one of the following types:



Recall that n and m must be even and odd in $\mathscr{B}$ and $\mathscr{C}$ respectively.

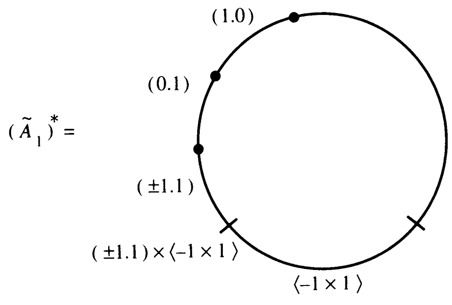
As in §2, we break the orbit space of each type into two pieces and investigate them as follows:

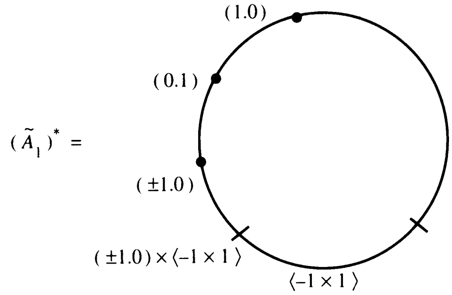
(i) $\mathscr{A}$ type



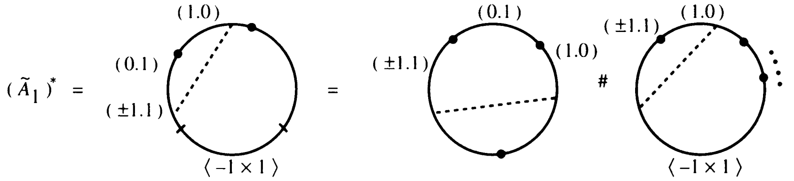
where A_1 and A_2 are 4-manifolds.

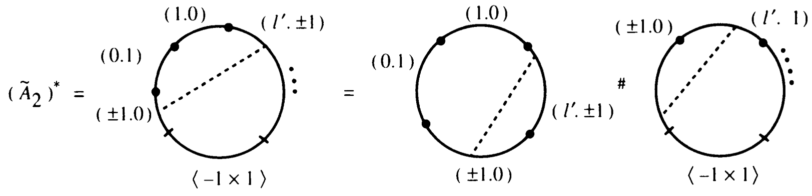
Then, by applying the same methods as in (3.2), M is diffeomorphic to a manifold $\widetilde{A}_i$ $(i = 1$ or $2)$ with orbit spaces



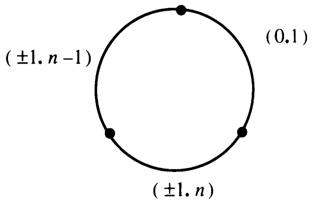


according as l is odd or even respectively. We can reduce the number of fixed points i.e. we can express $\widetilde{A}_i$ as a connected sum of a simply-connected manifold and some manifold K whose number of fixed points is less than that of $\widetilde{A}_i$ as shown below

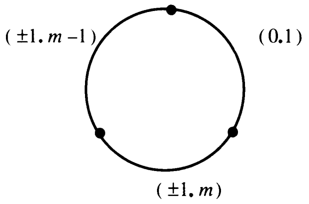




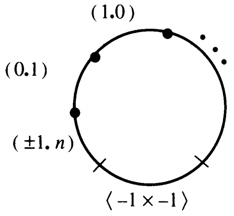
If we get type $\mathscr{A}$, then we apply the above method again. Suppose K_i is of type $\mathscr{B}$ or $\mathscr{C}$. By adding



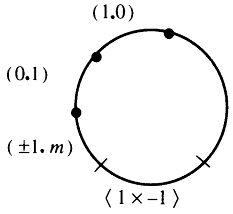
and



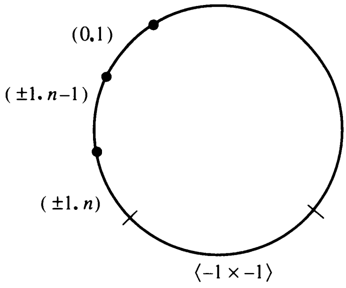
to

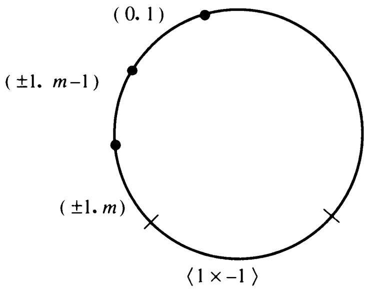


and



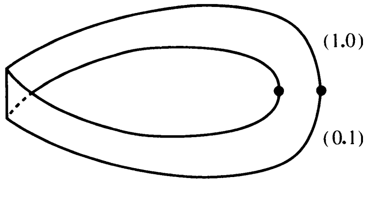
respectively, we have





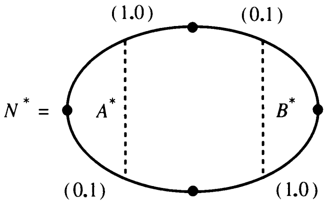
Then $\langle -1 \times -1\rangle$ is contained in $(1.n-1)$, and $\langle 1 \times -1\rangle$ is contained in $(1.m-1)$. By an automorphism of T^2, we get type $\mathscr{A}$ and apply the above argument until we get less than three fixed points. By (3.3) and (3.5), we obtain the desired result. □

(4.2) **Lemma.** *Suppose M^* is given as follows*:



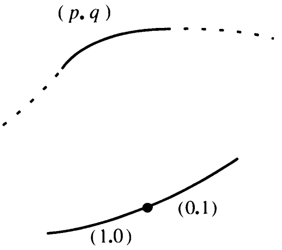
Then M *is diffeomorphic to* $S^2 \times S^2 \#(S^1 \times S^3/\simeq)$ *where* $\simeq$ *means* $(\alpha, (x, y, z, w))$ *is identified with* $(-\alpha, (x, y, z, -w))$.

Proof. M^* can be obtained as follows: Consider N^* as shown below:

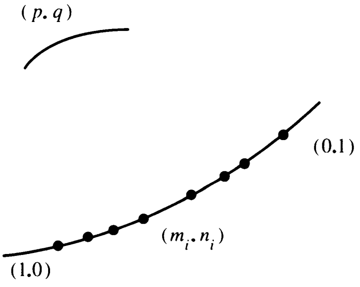


Note that $N = S^2 \times S^2$. Delete A^* and B^* whose preimage are 4-dimensional disks and attach ∂A^* and ∂B^* by using an orientation preserving diffeomorphism. So, in the total spaces, two boundaries of C are S^3. If we identify S^3 with the other S^3 by using an orientation preserving map f from S^3 to S^3, we get M. Thus $M = S^2 \times S^2 \#(S^3 \times S^1/\simeq)$.

(4.3) **Lemma.** *Suppose we have the following orbit space*:



where neither p *nor* q *are* ± 1. *By adding some number of* $\mathbb{C}P^2$*'s or* $\overline{\mathbb{C}P^2}$*'s to the total space, we can make the resulting orbit space as follows*:

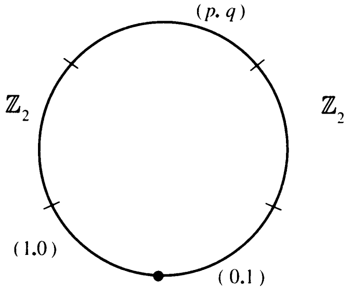


such that, for some $m_i n_i$,

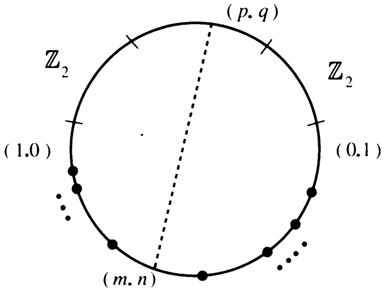
$$\begin{vmatrix} m_i & n_i \\ p & q \end{vmatrix} = 1 \quad or \; -1$$

Proof. This can be proved easily by using the Euclidean algorithm. □

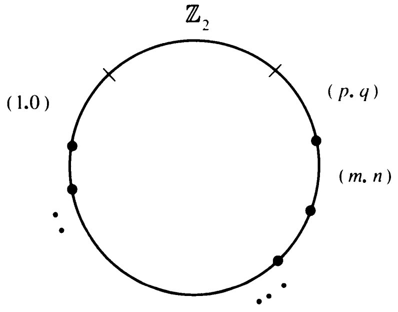
Remark. Suppose M^* is as shown below:

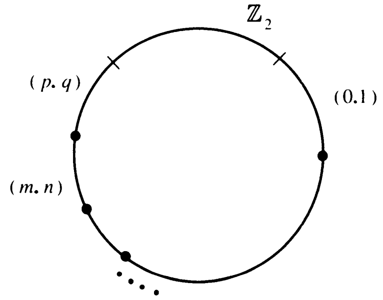


Then, by using (4.3), we get



such that $\left|\begin{smallmatrix} p & q \\ m & n \end{smallmatrix}\right| = \pm 1$. We decompose this into two orbit spaces



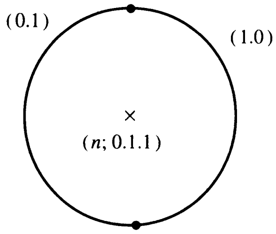


so that each of them has only one $\mathbb{Z}_2$ component.

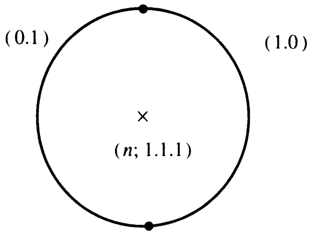
We are ready to prove the stable diffeomorphism theorem.

(4.4) **Theorem.** *Suppose* M^* *has no* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *stabilizer on each of the boundary and has a fixed point. Then* $M \# n\mathbb{C}P^2$ *can be expressed as a connected sum of* $S^2 \times S^2$, $\mathbb{C}P^2$, $RP^2 \times S^2$, $S^1 \times S^3/\simeq$, L_n, L'_n, $S^1 \times S^3$, K, *where* n *is some positive integer,* L_n, L'_n *are* 4*-manifolds whose orbit spaces are* (*respectively*)

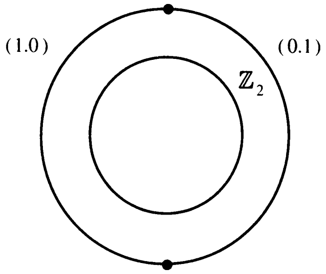
$L_n^* =$



$(L_n')^* =$



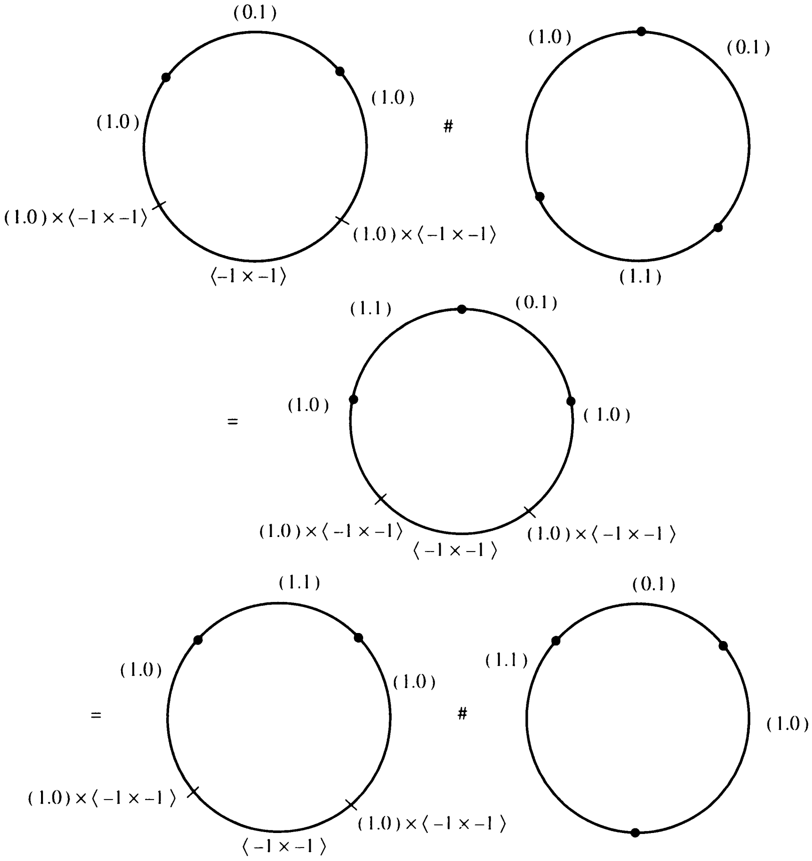
$K^* =$



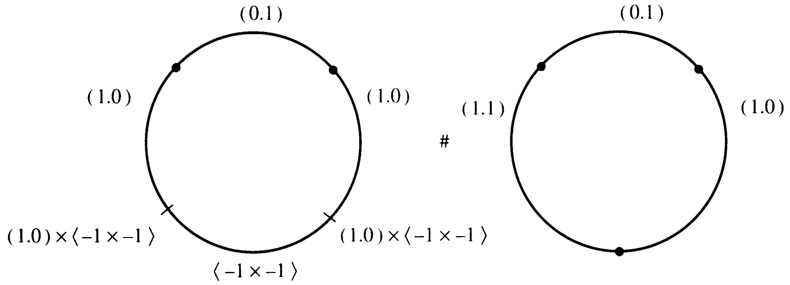
(*see* [P] *for more details*).

Proof. Since M^* has a fixed point, M^* is a 2-dimensional manifold with nonempty boundary. The proof follows from Theorem VI.1 of [P] and (2.7). □

Remark.



By reparametrizing T^2, we get



Thus $S^2 \times S^2/\simeq \#\mathbb{C}P^2 = RP^2 \times S^2\#\mathbb{C}P^2$, even if $S^2 \times S^2/\simeq$ is not homotopy equivalent to $RP^2 \times S^2$. In the same way, we can obtain $M_2\#\mathbb{C}P^2 = RP^2 \times S^2\#\mathbb{C}P^2$.

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